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ABSTRACT

The knowledge of a complete set of vector modes in cylindrical waveguides containing inhomogeneous media not necessarily bounded by perfect conductors permits the transformation of Maxwell's field equations into ordinary transmission-line equations. The determination of the electromagnetic fields produced by arbitrary currents in such regions is thereby reduced to a conventional network problem.

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1. Introduction

There are a number of common aspects in the electromagnetic problems that arise in connection with:

a) wave propagation along dielectric coated surfaces in " open" or "closed" waveguides;

b) tropospheric or duct propagation in planar (or cylindrical) atmospheres;

c) in general, wave propagation along cylindrical waveguides that contain inhomogeneous media and are not necessarily bounded by perfect conductors [1]

We shall be concerned with the determination of a complete set of vector modes in terms of which the electromagnetic fields produced by prescribed (or induced) sources in such inhomogeneous regions can be readily evaluated. The desired modes are defined by a vector eigenvalue problem and possess orthogonality properties that permit the reduction of a general source-excited vector field problem to a conventional scalar transmission-line problem.

In the steady state of time dependence $e^{-i\omega t}$, the electric field $\underset{\sim}{E}$ and the magnetic field $\underset{\sim}{H}$ produced by sources of electric current density $\underset{\sim}{J}$ and magnetic current density $\underset{\sim}{M}$ obey the field equations

$$\text{(1.1a)} \qquad \begin{aligned} \nabla \times \underset{\sim}{E} - ik\mu\, \underset{\sim}{H} &= -\underset{\sim}{M}\,, \\ \nabla \times \underset{\sim}{H} + ik\varepsilon\, \underset{\sim}{E} &= \underset{\sim}{J}\,, \end{aligned}$$

where $k = \omega\sqrt{\mu_o \varepsilon_o}$. In the cylindrical regions of interest, ε and μ, the dielectric constant and permeability relative to the vacuum values ε_o and μ_o, are assumed

to be independent of the axial coordinate z but are in general variable over the plane transverse to the z-axis. On the guide walls (if present) the fields are to be subject to the boundary conditions

$$\tilde{E} = \mathcal{Z} \cdot \tilde{H} \times \tilde{\nu} , \tag{1.1b}$$

where $\mathcal{Z}$ is an impedance dyadic relating the electric and magnetic field components tangential to the guide walls. As noted in Fig. 1, $\tilde{\nu}$ is the outer unit normal perpendicular to the $\tilde{z}_o$-axis and the guide walls $\tilde{s}$. All field quantities are r.m.s. and the normalization is such that the intrinsic impedance of vacuum is unity*.

The explicit solution of Eqs. (1) in cylindrical regions may be effected by a representation of the fields in terms of steady-state modes. The latter comprise the possible guided waves that can be propagated along the axis of such regions with no sources present. The field components of these modes are expressible in terms of vector eigenfunctions $\tilde{e}_a(\rho)$ and $\tilde{h}_a(\rho)$ which indicate the $(x,y) \equiv \tilde{\rho}$-dependence of the source-free electric and magnetic fields transverse to the transmission direction z. Since the eigenfunctions form a complete set, the total transverse electric and magnetic fields, which are independent field variables in Eqs. (1), may be represented at any point $(x,y,z) \equiv \tilde{r}$ as

$$\begin{aligned} \tilde{E}_t(\tilde{r}) &= \textstyle\sum_a V_a(z)\, \tilde{e}_a(\tilde{\rho}) \\ \tilde{H}_t(\tilde{r}) &= \textstyle\sum_a I_a(z)\, \tilde{h}_a(\tilde{\rho}) \end{aligned} \quad , \tag{1.2}$$

the summation being extended over the complete set of modes. With a proper determination of the mode functions $\tilde{e}_a$ and $\tilde{h}_a$ the original vector field equations (1) may be transformed by (2) into ordinary scalar differential equations for the mode amplitudes $V_a(z)$ and $I_a(z)$:

---

* For the conventional normalization replace $\tilde{H}$ and $\tilde{J}$ in (1) by

$$\sqrt{\frac{\mu_o}{\varepsilon_o}}\ \tilde{H} \quad \text{and} \quad \sqrt{\frac{\mu_o}{\varepsilon_o}}\ \tilde{J}.$$

$$
\begin{aligned}
\frac{dV_a}{dz} &= i\varkappa_a Z_a I_a - v_a, \\
&\qquad\qquad\qquad\qquad (1.3) \\
\frac{dI_a}{dz} &= i\varkappa_a Y_a V_a - i_a,
\end{aligned}
$$

where the constant parameters $\varkappa_a$, $Z_a = 1/Y_a$ and the z-dependent source functions v_a, i_a are known. Equations (3) pose a conventional transmission-line problem representative of Eqs. (1a) with the transverse spatial dependence of the typical mode a' suppressed. This problem is readily solved for the (voltage) amplitude V_a and the (current) amplitude I_a, and from these and Eqs. (2) the desired representation of the excited electromagnetic field is obtained. It should be noted, however, that the ability to obtain an explicit field solution is predicated upon the explicit evaluation of the mode functions $\underset{\sim}{e}_a$ and $\underset{\sim}{h}_a$ which can be realized analytically only in cylindrical regions with sufficiently simple ε, μ-variability.

The cylindrical region illustrated in cross-section in Fig. 1 represents the general uniform waveguide; the transmission direction of interest is along z.

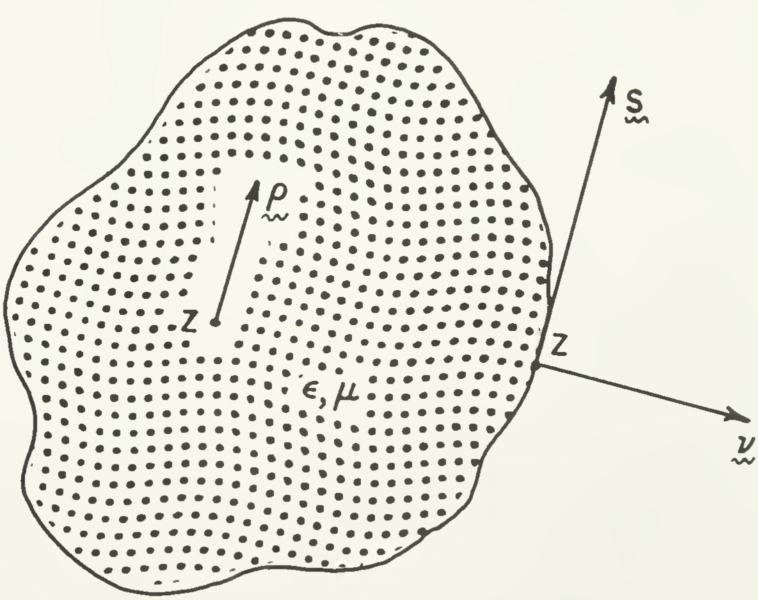

Fig. 1. Uniform waveguide with variable medium ε, and arbitrary walls (if any)

A special case to be treated in detail is the planar stratified region depicted in Fig. 2a. Although pictured as a region within which the dielectric

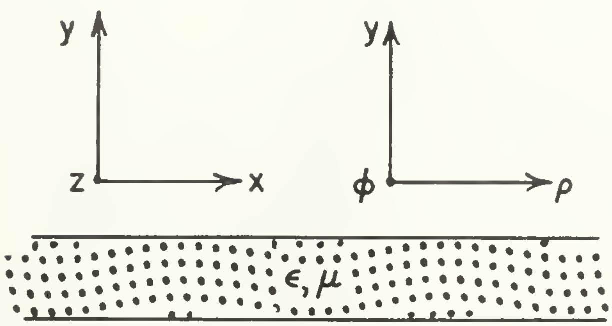

Fig. 2a. Open waveguide with planar stratification.

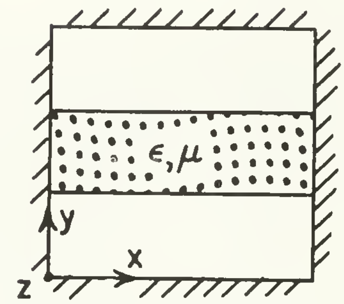

Fig. 2b. Closed waveguide with planar stratification.

constant ε and the permeability μ are piecewise constant, both ε and μ are permitted to have a relatively arbitrary variation along the y-direction of stratification. A region of this type is frequently termed a composite open waveguide. On introduction of a cartesian coordinate system, such a region may be regarded as a uniform waveguide with transmission direction along either the x, the y, or the z axis. On use of a polar coordinate system the region may be regarded as a uniform waveguide with transmission along the y-axis, as a non-uniform radial waveguide with transmission in the ρ-direction, or as an angular waveguide with transmission in the ϕ-direction. If the region is enclosed by planar walls, as for example in Fig. 2b, it becomes a closed composite waveguide of rectangular cross section. The boundary walls need not be assumed to be perfect conductors and will be regarded in general as 'impedance walls' in a manner defined more precisely below. It is of interest to investigate the circumstances under which the open region of Fig. 2a is a limit case of the closed region of Fig. 2b as the walls recede to infinity since the determination of a complete set of modes for a closed region is a relatively elementary problem.

The proper determination of a complete set of vector modes in the above regions poses a vector eigenvalue problem in the cross-section transverse to the transmission direction. The spectrum of modes obtained is usually discrete in the case of closed regions, as in Fig. 2b, and generally both discrete and continuous in the case of open regions, as in Fig. 2a. The discrete modes characterize the natural (free) cross-sectional resonances of a region in that they are the possible source-free field solutions each satisfying the prescribed boundary conditions in the cross-section; they constitute the proper eigenfunctions of the spectrum. On the other hand the continuous modes, which characterize the radiative part of the spectrum, individually are possible field solutions, but only as a complex (a wave-packet) do they satisfy the required cross-sectional boundary conditions; they constitute the improper eigenfunctions of the spectrum. Of the discrete modes encountered in both open and closed regions those characterized by an exponentially decreasing behavior outside the dielectric regions are designated as <u>surface waves</u>. They are further characterized by a relative insensitivity to the presence or absence of guide walls in the dielectric vicinity, i.e. to whether the guide cross-section is closed or open.

The discrete and continuous vector modes possess orthogonality properties over the cross-section that are determined by the nature of the dielectric medium and the guide walls, if any. In the <u>hermitean</u> case with non-dissipative dielectric and 'reactive' walls, the orthogonality properties imply no power coupling between modes, i.e. the total power in the field is the sum of the powers carried by the steady-state modes. In the <u>non-hermitean</u> case with dissipative dielectric and 'impedance' walls, there is power coupling between modes despite the orthogonality properties.

In this study our concern will be both with the determination of the complete set of modes in a fairly general uniform guide with variable ε, μ, and

with the concomittant transformation of the general field problem therein into a transmission-line problem. In this report the eigenvalue problem for the modes will be stated in general. Explicit evaluations of the mode functions will be performed for planar stratified regions wherein the vector eigenvalue problem is separable in a subsequent report (Part II). The methods employed are a generalization and extension of those reported previously[2].

2. Eigenvalue Problem for Modes

It is of interest to ascertain whether or not the possible field solutions in a source-free but otherwise arbitrary uniform guide suffice, when suitably superposed, to represent the field when arbitrary sources are present. In the absence of sources the possible field solutions, each satisfying prescribed boundary conditions on the guided walls, are designated as proper modes. From general symmetry properties of a uniform guide one can infer about such modes a number of characteristics that are independent of the specific nature of the guide cross-section and hence are applicable to a large class of uniform waveguides.

The distinguishing symmetry property of a source-free cylindrical guide is its invariance to translation along the guide axis. The independence of the properties of the medium and walls of a uniform guide on the axial coordinate z implies a field dependence characteristic of the translation operator $\frac{1}{i}\,\frac{\partial}{\partial z}$. The eigenfunctions of the translation operator have the form $e^{i\varkappa z}$. Accordingly, the fields of a possible mode in a uniform guide will likewise have a separable $e^{i\varkappa z}$-dependence, with $\varkappa$, the mode wave number, to be determined. The steady-state electric and magnetic fields of a typical mode are thus given to within an amplitude factor by

$$\underset{\sim}{E}(\underset{\sim}{r}) = \underset{\sim}{\mathscr{E}}(\underset{\sim}{\rho})\, e^{i\varkappa z}, \qquad (2.1)$$

$$\underset{\sim}{H}(\underset{\sim}{r}) = \underset{\sim}{\mathscr{H}}(\underset{\sim}{\rho})\, e^{i\varkappa z},$$

which represents a guided wave of invariable cross-sectional form traveling in the direction of increasing z. Further information is derived if the guide is also invariant to a reflection in any cross-sectional plane. This reflection symmetry implies that under a transformation $z \rightarrow -z$, the transverse (t) and longitudinal (z) components of the mode fields transform to within a phase factor as

$$(2.2) \qquad \begin{aligned} &\underset{\sim}{E}_t(\underset{\sim}{\rho},z) \rightarrow \underset{\sim}{E}_t(\underset{\sim}{\rho}, -z), \qquad && \underset{\sim}{H}_t(\underset{\sim}{\rho},z) \rightarrow -\underset{\sim}{H}_t(\underset{\sim}{\rho}, -z), \\ &E_z(\underset{\sim}{\rho},z) \rightarrow -E_z(\underset{\sim}{\rho}, -z), && H_z(\underset{\sim}{\rho},z) \rightarrow H_z(\underset{\sim}{\rho}, z) . \end{aligned}$$

Thus if Eqs. (1) provide a mode solution of the source-free field equations, a 'reflected mode' with wave number $-\varkappa$ and with field components related by (2) to those in Eqs. (1) likewise provides a field solution.

The vector mode functions $\underset{\sim}{\mathcal{E}}(\underset{\sim}{\rho})$ and $\underset{\sim}{\mathcal{H}}(\underset{\sim}{\rho})$ together with $\varkappa$ are to be so determined that (1) provides a source-free solution of the field equations (1.1), i.e.,

$$(2.2a) \qquad \begin{aligned} \nabla \times \underset{\sim}{\mathcal{E}} &= ik\mu \, \underset{\sim}{\mathcal{H}}, \\ \nabla \times \underset{\sim}{\mathcal{H}} &= -ik\varepsilon \, \underset{\sim}{\mathcal{E}} , \end{aligned}$$

where $\varepsilon = \varepsilon(\underset{\sim}{\rho})$ and $\mu = \mu(\underset{\sim}{\rho})$ and $\nabla = \nabla_t + i\varkappa \underset{\sim}{z}_o$; $\underset{\sim}{z}_o$ denotes the unit axial vector and ∇_t the transverse gradient operator. In accordance with (1.1b) the discrete mode fields are to be subject for all z to the boundary condition

$$(2.2b) \qquad \underset{\sim}{\mathcal{E}} = \mathcal{Z} \cdot \underset{\sim}{\mathcal{H}} \times \nu$$

on the guide walls $\underset{\sim}{s}$. In uniform guides with reflection symmetry the impedance dyadic $\mathcal{Z}$ is diagonal in a $\underset{\sim}{s}$, $\underset{\sim}{z}_o$ basis.

An alternative form of Eqs. (2) is obtained by decomposition into components transverse and longitudinal to z (cf. ref. [2]). The transverse component ob-

tained by vector product multiplication of Eqs. (2a) by $\underset{\sim}{z}_o$ is

$$\nabla_t \mathcal{E}_z - i\varkappa \underset{\sim}{\mathcal{E}}_t = -ik\mu \underset{\sim}{\mathcal{H}}_t \times \underset{\sim}{z}_o \tag{2.3a}$$
$$\nabla_t \mathcal{H}_z - i\varkappa \underset{\sim}{\mathcal{H}}_t = -ik\varepsilon\, \underset{\sim}{z}_o \times \underset{\sim}{\mathcal{E}}_t$$

and the longitudinal component obtained by scalar product multiplication of Eqs. (2a) by $\underset{\sim}{z}_o$ is

$$\nabla_t \cdot \underset{\sim}{z}_o \times \underset{\sim}{\mathcal{E}}_t = -ik\mu\, \mathcal{H}_z \tag{2.3b}$$
$$\nabla_t \cdot \underset{\sim}{\mathcal{H}}_t \times \underset{\sim}{z}_o = -ik\varepsilon\, \mathcal{E}_z \; .$$

On elimination of the longitudinal components in (3a) by use of (3b) one obtains as the defining equation for the transverse components

$$\varkappa \underset{\sim}{\mathcal{E}}_t = k \left(\mu\, 1_t + \frac{1}{k^2} \nabla_t \frac{1}{\varepsilon} \nabla_t\right) \cdot \underset{\sim}{\mathcal{H}}_t \times \underset{\sim}{z}_o \tag{2.4a}$$
$$\varkappa \underset{\sim}{\mathcal{H}}_t = k \left(\varepsilon\, 1_t + \frac{1}{k^2} \nabla_t \frac{1}{\mu} \nabla_t\right) \cdot \underset{\sim}{z}_o \times \underset{\sim}{\mathcal{E}}_t \, ,$$

where 1_t is the transverse unit dyadic. In view of Eqs. (2b) and (3b), the discrete mode fields of Eqs. (4a) are to be subject to the boundary conditions

$$\nabla_t \cdot \underset{\sim}{z}_o \times \underset{\sim}{\mathcal{E}}_t = ik\mu\, Z_{ss}^{-1}(s)\, \underset{\sim}{z}_o \times \underset{\sim}{\mathcal{E}}_t \cdot \underset{\sim}{\nu} \tag{2.4b}$$
$$\nabla_t \cdot \underset{\sim}{\mathcal{H}}_t \times \underset{\sim}{z}_o = ik\varepsilon\, Z_{zz}(s) \underset{\sim}{\mathcal{H}}_t \times \underset{\sim}{z}_o \cdot \underset{\sim}{\nu}$$

on the guide walls $\underset{\sim}{s}$, Z_{ss} and Z_{zz} being the $\varkappa$-independent elements of the diagonal dyadic $\mathcal{Z}$. Equations (2) or their transverse equivalent, Eqs. (4), constitute a vector eigenvalue problem for the determination both of the possible mode fields $\underset{\sim}{\mathcal{E}}_t$, $\underset{\sim}{\mathcal{H}}_t$ and of the possible wave numbers $\varkappa$ in a uniform region with

variable ε, μ medium. The conventional eigenvalue problem for the modes in a homogeneous cylindrical guide with perfectly conducting walls is a special case of Eqs. (4) in which $\varepsilon = 1 = \mu$ and $\mathcal{J} = 0$. In the present case the possible mode vectors $\mathcal{E}_t$ and $\mathcal{H}_t$ need not be perpendicular as in the special case.

The reciprocity properties of the electromagnetic field permit a ready derivation of the orthogonality properties of the mode fields (1)†. Two cases are to be distinguished: the hermitean and the non-hermitean. In the hermitean case the dielectric medium and guide walls are non-dissipative, whence ε, μ, and $i\mathcal{J}$ are real. In this case it can be immediately inferred from the constancy of the real power flow, $\mathrm{Re}\,\mathcal{E}\times\mathcal{H}^* \cdot z_o$, along the guide axis that the possible squared wave numbers $\varkappa^2$ of the mode fields (1) must be real. If two such modes are distinguished by subscripts a and b, the complex reciprocity theorem for source-free fields

$$\nabla \cdot (E_a \times H_b^* + E_b^* \times H_a) = 0 \tag{2.5}$$

implies that

$$\nabla_t \cdot (\mathcal{E}_a \times \mathcal{H}_b^* + \mathcal{E}_b^* \times \mathcal{H}_a) + i(\varkappa_a - \varkappa_b^*)\, z_o \cdot (\mathcal{E}_a \times \mathcal{H}_b^* + \mathcal{E}_b^* \times \mathcal{H}_a) = 0.$$

On integration over the guide cross-section S and use of the two-dimensional divergence theorem††, one obtains, in view of Eqs. (2b) and the realness of the diagonal dyadic $i\mathcal{J}$,

$$i(\varkappa_a - \varkappa_b^*) \iint_S (\mathcal{E}_a \cdot \mathcal{H}_b^* \times z_o + \mathcal{H}_a \cdot z_o \times \mathcal{E}_b^*)\, dS$$
$$= -\oint_s \left\{ \mathcal{H}_b^* \times \nu \cdot (\mathcal{E}_a - \mathcal{J} \cdot \mathcal{H}_a \times \nu) + \mathcal{H}_a \times \nu \cdot (\mathcal{E}_b - \mathcal{J} \cdot \mathcal{H}_b \times \nu)^* \right\} ds = 0$$

---

† The following discussion of orthogonality properties is similar to that given by Adler[1] and is included because of its relevance in the anisotropic cases to be discussed in a separate report (Part III).

†† $\iint_S \nabla_t \cdot A_t \, dS = \int \nu \cdot A_t \, ds$ when A is suitably continuous and the line integral with respect to s is taken over the periphery s bounding S.

whence by Eqs. (2b) one has the bi-orthogonality property

(2.6a) $$\iint_S (\underset{\sim}{\mathscr{E}}_a \cdot \underset{\sim}{\mathscr{H}}_b^* \times \underset{\sim}{z}_o + \underset{\sim}{\mathscr{H}}_a \cdot \underset{\sim}{z}_o \times \underset{\sim}{\mathscr{E}}_b^*)\, dS = 0 \quad \text{if } \varkappa_a \neq \varkappa_b^*.$$

In addition, if instead of the mode a the 'reflected mode' $-a$ is used one obtains by (2) the orthogonality property

(2.6b) $$\iint_S (\underset{\sim}{\mathscr{E}}_a \cdot \underset{\sim}{\mathscr{H}}_b^* \times \underset{\sim}{z}_o - \underset{\sim}{\mathscr{H}}_a \cdot \underset{\sim}{z}_o \times \underset{\sim}{\mathscr{E}}_b^*)\, dS = 0 \quad \text{if } -\varkappa_a \neq \varkappa_b^*.$$

On addition of Eqs. (6a) and (6b), one has the simple bi-orthogonality property

(2.7) $$\iint_S \underset{\sim}{\mathscr{E}}_a \cdot \underset{\sim}{\mathscr{H}}_b^* \times \underset{\sim}{z}_o\, dS = 0 \qquad \text{if } \varkappa_a^2 \neq \varkappa_b^2.$$

In the non-hermitean case the dielectric medium and guide walls may be dissipative, in which event ε, μ, $\mathscr{Z}$ and consequently $\varkappa$ are in general complex. The orthogonality properties of any two source-free modes, distinguished by subscripts a and b, may now be inferred from the reciprocity theorem

(2.8) $$\nabla \cdot (\underset{\sim}{E}_a \times \underset{\sim}{H}_b - \underset{\sim}{E}_b \times \underset{\sim}{H}_a) = 0$$

in a manner similar to the development in Eqs. (5)-(7). For the mode fields defined by Eqs. (1) one derives

$$\iint_S (\underset{\sim}{\mathscr{E}}_a \cdot \underset{\sim}{\mathscr{H}}_b \times \underset{\sim}{z}_o - \underset{\sim}{\mathscr{H}}_a \cdot \underset{\sim}{z}_o \times \underset{\sim}{\mathscr{E}}_b)\, dS = 0 \quad \text{if } -\varkappa_a \neq \varkappa_b.$$

If the region possesses reflection symmetry, then

$$\iint_S (\underset{\sim}{\mathscr{E}}_a \cdot \underset{\sim}{\mathscr{H}}_b \times \underset{\sim}{z}_o + \underset{\sim}{\mathscr{H}}_a \cdot \underset{\sim}{z}_o \times \underset{\sim}{\mathscr{E}}_b)\, dS = 0 \quad \text{if } \varkappa_a \neq \varkappa_b,$$

whence in a cylindrical region with reflection symmetry

$$\iint_S \underset{\sim}{\mathscr{E}}_a \cdot \underset{\sim}{\mathscr{H}}_b \times \underset{\sim}{z}_o \, dS = 0 \qquad \text{if } \kappa_a^2 \neq \kappa_b^2 . \tag{2.9}$$

It is of interest to observe that since the orthogonality properties (7) and (9) involve only transverse field components, these properties may equally well be derived directly from the transverse eigenvalue problem of Eqs. (4).

The primary importance of the transverse field will be emphasized by introducing the notation

$$\underset{\sim}{\mathscr{E}}_t(\underset{\sim}{\rho}) = \underset{\sim}{e}(\underset{\sim}{\rho}) \quad \text{and} \quad \underset{\sim}{\mathscr{H}}_t(\underset{\sim}{\rho}) = Y \, \underset{\sim}{h}(\underset{\sim}{\rho}) \tag{2.10a}$$

for the transverse components of a mode field, the parameter $Y = 1/Z$ being introduced to secure a convenient normalization of $\underset{\sim}{e}$ relative to $\underset{\sim}{h}$. On substitution of (10a) the transverse eigenvalue problem (4a) can be restated in the form

$$\begin{aligned} \kappa Z \, \underset{\sim}{e} &= k(\mu \, 1_t + \frac{1}{k^2} \nabla_t \frac{1}{\varepsilon} \nabla_t) \cdot \underset{\sim}{h} \times \underset{\sim}{z}_o \\ \kappa Y \, \underset{\sim}{h} &= k(\varepsilon \, 1_t + \frac{1}{k^2} \nabla_t \frac{1}{\mu} \nabla_t) \cdot \underset{\sim}{z}_o \times \underset{\sim}{e} \end{aligned} \tag{2.10b}$$

subject to the boundary conditions that follow from (4b). As observed in (2), there exist both $+\kappa$ and $-\kappa$ eigensolutions of Eqs. (10b). If the parameter $+Y$ is associated with the $+\kappa$ solution and $-Y$ with the $-\kappa$ solution, then it is evident from either (2) or (10b) that the mode functions $\underset{\sim}{e}$, $\underset{\sim}{h}$ for the $\pm\kappa$ solutions become identical. In view of this identity as well as of the orthogonality properties (7) and (9), a mode will be distinguished hereafter not by the wave number κ but rather by κ^2. Such a mode is generally a standing wave formed by superposition of a wave traveling in the positive z-direction and its reflection traveling in the negative z-direction. The transverse field components of a

typical mode $\varkappa^2$ in a source-free region can thus be represented as

$$
\begin{aligned}
\underset{\sim}{E}_t(\underset{\sim}{r}) &= V(z)\, \underset{\sim}{e}(\underset{\sim}{\rho}) \\
\underset{\sim}{H}_t(\underset{\sim}{r}) &= I(z)\, \underset{\sim}{h}(\underset{\sim}{\rho}),
\end{aligned}
\tag{2.11}
$$

where by Eqs. (1) and (2) the z-dependent mode amplitudes V and I have the form

$$
\begin{aligned}
V(z) &= A_+ e^{i\varkappa z} + A_- e^{-i\varkappa z} \\
I(z) &= Y(A_+ e^{i\varkappa z} - A_- e^{-i\varkappa z}).
\end{aligned}
$$

In consequence of the arbitrariness of $A_{\pm}$ there is an arbitrariness in the mode amplitudes V, I, which is exhibited in a significant form by noting that V and I satisfy the first-order differential equations

$$
\begin{aligned}
\frac{dV}{dz} &= i\varkappa Z I \\
\frac{dI}{dz} &= i\varkappa Y V,
\end{aligned}
\tag{2.12}
$$

where $Z = 1/Y$. Since Eqs. (12) have the form of homogeneous transmission-line equations, one employs the conventional designations of V as the 'voltage' amplitude and I as the 'current' amplitude of the mode in question.

In an arbitrary cylindrical waveguide excited by sources, the ability to represent the electromagnetic fields as a superposition of source-free modes of the form (11) is predicated upon the completeness of the set of possible mode functions $\underset{\sim}{e}_a$, $\underset{\sim}{h}_a$. The completeness will be exhibited in the subsequent procedure employed to obtain explicit solutions of the vector eigenvalue problem (cf. Part II). The development to this point has indicated that if in a general source-free region possessing translation and reflection symmetry there exist possible mode functions $\underset{\sim}{e}$, $\underset{\sim}{h}$ defined by Eqs. (10b), they must possess orthogonality properties of the form (7) or (9).

3. Field Representations in Regions with Sources

The vector modes characteristic of a source-free region provide a representation of the electromagnetic field in the presence of sources. In this section we shall describe such a representation for a uniform waveguide with arbitrary cross-section, with arbitrary isotropic ε, μ cross-sectional variation, and with rather general boundary conditions at the guide walls (if any). The electromagnetic fields are determined by the inhomogeneous field Eqs. (1.1a) subject to the boundary conditions (1.1b) on the guide walls. As in Eqs. (2.2) - (2.4) one decomposes the inhomogeneous field equations into transverse components

$$
\begin{aligned}
&\nabla_t E_z - \frac{\partial}{\partial z} \underset{\sim}{E}_t = -ik\mu \underset{\sim}{H}_t \times \underset{\sim}{z}_o + \underset{\sim}{M} \times \underset{\sim}{z}_o \\
&\nabla_t H_z - \frac{\partial}{\partial z} \underset{\sim}{H}_t = -ik\varepsilon\, \underset{\sim}{z}_o \times \underset{\sim}{E}_t + \underset{\sim}{z}_o \times \underset{\sim}{J}
\end{aligned}
\tag{3.1}
$$

and longitudinal components

$$
\begin{aligned}
&\nabla_t \cdot \underset{\sim}{z}_o \times \underset{\sim}{E}_t = -ik\mu H_z + M_z \\
&\nabla_t \cdot \underset{\sim}{H}_t \times \underset{\sim}{z}_o = -ik\varepsilon E_z + J_z ,
\end{aligned}
\tag{3.2}
$$

whence on elimination of the longitudinal components from (1) one obtains as the defining equations for the transverse field components:

$$
\begin{aligned}
&\frac{\partial}{\partial z} \underset{\sim}{E}_t = ik \left(\mu\, 1_t + \frac{1}{k^2} \nabla_t \frac{1}{\varepsilon} \nabla_t\right) \cdot \underset{\sim}{H}_t \times \underset{\sim}{z}_o + \underset{\sim}{M} \times \underset{\sim}{z}_o + i\nabla_t \left(\frac{J_z}{k\varepsilon}\right) \\
&\frac{\partial}{\partial z} \underset{\sim}{H}_t = ik \left(\varepsilon\, 1_t + \frac{1}{k^2} \nabla_t \frac{1}{\mu} \nabla_t\right) \cdot \underset{\sim}{z}_o \times \underset{\sim}{E}_t + \underset{\sim}{z}_o \times \underset{\sim}{J} + i\nabla_t \left(\frac{M_z}{k\mu}\right),
\end{aligned}
\tag{3.3}
$$

subject to the boundary conditions (1.1b).

Since the vector modes defined in Section 2 can be shown to comprise a complete set, they suffice to describe the electromagnetic field in a guide containing sources. As the characteristic field patterns $\underset{\sim}{e}_a(\rho)$, $\underset{\sim}{h}_a(\underset{\sim}{\rho})$ of each

mode are known over the guide section, only the mode amplitude variation along the guide has to be determined. Accordingly, the transverse r.m.s. components of the electric and magnetic fields will be represented by the linear superposition

$$(3.4)\qquad \begin{aligned} \underset{\sim}{E}_t(\underset{\sim}{r}) &= \sum_a V_a(z)\, \underset{\sim}{e}_a(\underset{\sim}{\rho}) \\ \underset{\sim}{H}_t(\underset{\sim}{r}) &= \sum_a I_a(z)\, \underset{\sim}{h}_a(\underset{\sim}{\rho}); \end{aligned}$$

here and in the following the summation is to be extended over all possible values of the mode subscript $\underline{a}$ in the discrete and continuous spectrum.

a) Hermitean case

In the hermitean case wherein the guide has a non-dissipative medium and reactive walls, the mode functions $\underset{\sim}{e}_a$ and $\underset{\sim}{h}_a$ can be so normalized that their bi-orthgonality properties can be written in accord with Eq. (2.7) as

$$(3.5)\qquad \iint_S \underset{\sim}{e}_a \cdot \underset{\sim}{h}_b^* \times \underset{\sim}{z}_o \; dS = \delta_{ab} = \begin{cases} 0, & a \neq b \\ 1, & a = b, \end{cases}$$

the surface integrals being extended over the entire guide cross-section S. Eqs. (5) imply that the total complex power flow at any point z is given by

$$(3.6)\qquad P = \iint_S \underset{\sim}{E}_t \times \underset{\sim}{H}_t^* \cdot \underset{\sim}{z}_o \; dS = \sum_a V_a(z)\, I_a^*(z)$$

and is the sum of the powers carried by the individual modes. In view of the orthogonality properties (5) the mode amplitudes and the transverse field are related by the transform equations

$$(3.7)\qquad \begin{aligned} V_a(z) &= \iint_S \underset{\sim}{E}_t(\underset{\sim}{r}) \cdot \underset{\sim}{h}_a^*(\underset{\sim}{\rho}) \times \underset{\sim}{z}_o \; dS \\ I_a(z) &= \iint_S \underset{\sim}{H}_t(\underset{\sim}{r}) \cdot \underset{\sim}{z}_o \times \underset{\sim}{e}_a^*(\underset{\sim}{\rho}) \; dS \; . \end{aligned}$$

The explicit determination of the mode amplitudes V_a, I_a is accomplished by transformation of the transverse field equation (3) in accordance with the operations indicated in Eq. (7). Thus, integrating over the guide cross-section the scalar product of the first of Eqs. (3) with $\underset{\sim}{h}_a^* \times \underset{\sim}{z}_o$ and the second with $\underset{\sim}{z}_o \times \underset{\sim}{e}_a^*$, one finds by Eqs. (7) (cf. ref. [2])

$$\text{(3.8a)}\qquad \begin{aligned} \frac{dV_a}{dz} &= ik \iint \underset{\sim}{h}_a^* \times \underset{\sim}{z}_o \cdot \left(\mu\, 1_t + \frac{1}{k^2}\, \nabla_t \frac{1}{\varepsilon} \nabla_t\right) \cdot \underset{\sim}{H}_t \times \underset{\sim}{z}_o \, dS + v_a(z) \\ \frac{dI_a}{dz} &= ik \iint \underset{\sim}{z}_o \times \underset{\sim}{e}_a^* \cdot \left(\varepsilon\, 1_t + \frac{1}{k^2}\, \nabla_t \frac{1}{\mu} \nabla_t\right) \cdot \underset{\sim}{z}_o \times \underset{\sim}{E}_t \, dS + i_a(z) \end{aligned}$$

where †

$$\text{(3.8b)}\qquad \begin{aligned} v_a(z) &= \iint_S \underset{\sim}{M}(\underset{\sim}{r}) \cdot \underset{\sim}{h}_a^*(\underset{\sim}{\rho})\, dS + i \iint_S \nabla_t \left(\frac{J_z}{k\varepsilon}\right) \cdot \underset{\sim}{h}_a^* \times \underset{\sim}{z}_o \, dS \\ i_a(z) &= \iint_S \underset{\sim}{J}(\underset{\sim}{r}) \cdot \underset{\sim}{e}_a^*(\underset{\sim}{\rho})\, dS + i \iint_S \nabla_t \left(\frac{M_z}{k\mu}\right) \cdot \underset{\sim}{z}_o \times \underset{\sim}{e}_a^* \, dS . \end{aligned}$$

The integrals on the right-hand side of Eqs. (8a) may be simplified on use of integration by parts, which gives the adjointness relation

$$\text{(3.9)}\qquad \begin{aligned} &\iint_S \left[\left(\nabla_t \frac{1}{\varepsilon} \nabla_t \cdot \underset{\sim}{H}_t \times \underset{\sim}{z}_o\right) \cdot \underset{\sim}{h}_a^* \times \underset{\sim}{z}_o - \underset{\sim}{H}_t \times \underset{\sim}{z}_o \cdot \left(\nabla_t \frac{1}{\varepsilon} \nabla_t \cdot \underset{\sim}{h}_a^* \times \underset{\sim}{z}_o\right)\right] dS \\ &= \int_s \frac{1}{\varepsilon} \left[\left(\underset{\sim}{\nu} \cdot \underset{\sim}{h}_a^* \times \underset{\sim}{z}_o\right) \nabla_t \cdot \underset{\sim}{H}_t \times \underset{\sim}{z}_o - \left(\underset{\sim}{\nu} \cdot \underset{\sim}{H}_t \times \underset{\sim}{z}_o\right) \nabla_t \cdot \underset{\sim}{h}_a^* \times \underset{\sim}{z}_o\right] ds \end{aligned}$$

and the dual relation obtained by the replacement of $\underset{\sim}{H}_t$, $\underset{\sim}{h}_a$, ε by $\underset{\sim}{E}_t$, $\underset{\sim}{e}_a$, μ, respectively; the surface integrals over the guide cross-section S are expressed in terms of line integrals over the peripheral curve s bounding the cross-section.

† The quantity i_a is of course not to be confused with the symbol i for $\sqrt{-1}$.

In view of the defining Eqs. (2.10b) for the mode functions $\underset{\sim}{e}_a$, $\underset{\sim}{h}_a$ and the boundary conditions (1.1b), the line integrals in Eqs. (9) vanish and the transformed field equations (8a) become by Eqs. (7)

$$
\begin{aligned}
\frac{dV_a}{dz} &= i\varkappa_a Z_a I_a - v_a \\
\frac{dI_a}{dz} &= i\varkappa_a Y_a V_a - i_a .
\end{aligned} \tag{3.10}
$$

The introduction of the mode representation (4) has transformed the original vector field problem posed by Eqs. (1.1a), (1.1b) into the 'a-dependent' scalar differential equations (10). The latter are conventional inhomogeneous transmission-line equations of the same nature as those encountered in the mode treatment of homogeneous perfectly conducting uniform guides. The determination of the behavior of the voltage V_a and current I_a is a straightforward network problem of finding the response to the known 'source voltage', v_a, and 'source current', i_a. With the evaluation of V_a and I_a from Eqs. (10) the electromagnetic field at any point $\underset{\sim}{r}$ can be synthesized by use of Eqs. (4).

The knowledge of the dependence of the $\varkappa_a$, Z_a, v_a, and i_a on the structure of the uniform guide, and on the frequency, the excitation, etc., is a necessary prerequisite to the solution of Eqs. (10). The quantities $\varkappa_a$ and Z_a are determined from the solution of the eigenvalue problem. The dependence of v_a and i_a on the excitation $\underset{\sim}{J}$ and $\underset{\sim}{M}$ can be cast in a form more general than that of Eqs. (8b) since the latter assumes differentiability of J_z/ε and M_z/μ over the guide cross-section. The desired form is obtained on integrating the right-hand integrals in (8b) by parts and assuming the J_z vanishes on the guide periphery; one finds

$$v_a(z) = \iint_S \underset{\sim}{M}(\underset{\sim}{r}) \cdot \underset{\sim}{h}_a^*(\underset{\sim}{\rho})\, dS + Z_a \iint_S \underset{\sim}{J}(\underset{\sim}{r}) \cdot \underset{\sim}{e}_{za}^*(\underset{\sim}{\rho})\, dS$$

(3.11)

$$i_a(z) = \iint_S \underset{\sim}{J}(\underset{\sim}{r}) \cdot \underset{\sim}{e}_a^*(\underset{\sim}{\rho})\, dS + Y_a \iint_S \underset{\sim}{M}(\underset{\sim}{r}) \cdot \underset{\sim}{h}_{za}^*(\underset{\sim}{\rho})\, dS,$$

where by definition

$$Z_a\, \underset{\sim}{e}_{za} = - \frac{\nabla_t \cdot \underset{\sim}{h}_a \times \underset{\sim}{z}_o}{iks}\, \underset{\sim}{z}_o$$

$$Y_a\, \underset{\sim}{h}_{za} = - \frac{\nabla_t \cdot \underset{\sim}{z}_o \times \underset{\sim}{e}_a}{ik\mu}\, \underset{\sim}{z}_o \quad .$$

The expressions (11) for v_a and i_a, which are applicable even for discontinuous J_z/ε and M_z/μ, are fundamental to the network calculation of the fields produced by arbitrary excitation $\underset{\sim}{J}$ and $\underset{\sim}{M}$ in an arbitrary cylindrical guide.

b) Non-Hermitean case.

When the medium and guide walls are dissipative, the vector mode functions $\underset{\sim}{e}_a$ and $\underset{\sim}{h}_a$ will be so normalized that their bi-orthogonality properties may be written as

(3.12)
$$\iint_S \underset{\sim}{e}_a \cdot \underset{\sim}{h}_b \times \underset{\sim}{z}_o\, dS = \delta_{ab}$$

in acccord with Eqs. (2.9). In contrast to the hermitean case the complex power flow cannot be expressed as the sum of the complex powers carried by the individual modes. Nevertheless the treatment of this case parallels that presented in Eqs. (5)-(10) for the hermitean case provided only that the complex conjugate, designated by the asterisk, is omitted in these equations. With this omission, the transmission-line equations (10) are still applicable in this case except that v_a and i_a are determined by

$$v_a(z) = \iint_S \underset{\sim}{M}(\underset{\sim}{r}) \cdot \underset{\sim}{h}_a(\underset{\sim}{\rho})\, dS - Z_a \iint_S \underset{\sim}{J}(\underset{\sim}{r}) \cdot \underset{\sim}{e}_{za}(\underset{\sim}{\rho})\, dS,$$

(3.13)

$$i_a(z) = \iint_S \underset{\sim}{J}(\underset{\sim}{r}) \cdot \underset{\sim}{e}_a(\underset{\sim}{\rho})\, dS - Y_a \iint_S \underset{\sim}{M}(\underset{\sim}{r}) \cdot \underset{\sim}{h}_{za}(\underset{\sim}{\rho})\, dS,$$

which differ from (11) only in that the sign of the last term is negative, since e_z and h_z are imaginary.

4. Planar Stratified Regions

The determination of the modes in an arbitrary uniform waveguide with axis along the z-direction and with variable ε and μ in the cross-section transverse thereto leads to a vector eigenvalue problem which is in general difficult to solve. For the special case of a planar stratified cross-section (cf. Fig. 2), wherein ε and μ are scalar functions of only one transverse coordinate y, the vector eigenvalue problem can be separated into two scalar eigenvalue problems. This separability is not unexpected, for such a region can be regarded not only as a guide with transmission axis along z but also, for the purpose of mode determination, as a simple terminated uniform guide with axis along y. When the functional form of $\varepsilon(y)$ and $\mu(y)$ is sufficiently simple, explicit expressions for the desired mode functions can be obtained by evaluation of the transverse resonances in this terminated waveguide. The latter are defined by a general Sturm-Liouville problem that admits both a discrete and continuous spectrum and will be evaluated explicitly in another report (Part II) for a number of typical cases.

The vector eigenvalue problem defined in Eqs. (2.2) can be decomposed into components transverse and longitudinal to the y-direction of stratification. Then by a development similar to that of Eqs. (2.3)-(2.4) one obtains as vector eigenvalue equations for the desired mode fields transverse (T) to the y-axis, that is, for $\underset{\sim}{\mathcal{E}}_T(x,y)\, e^{i\kappa z}$ and $\underset{\sim}{\mathcal{H}}_T(x,y)\, e^{i\kappa z}$,

(4.1a)
$$\frac{\partial \underset{\sim}{\mathcal{E}}_T}{\partial y} = ik\,(\mu\, 1_T + \frac{1}{k^2\varepsilon}\nabla_T\nabla_T)\cdot \underset{\sim}{\mathcal{H}}_T \times \underset{\sim}{y}_o$$
$$\frac{\partial \underset{\sim}{\mathcal{H}}_T}{\partial y} = ik\,(\varepsilon\, 1_T + \frac{1}{k^2\mu}\nabla_T\nabla_T)\cdot \underset{\sim}{y}_o \times \underset{\sim}{\mathcal{E}}_T\,;$$

with the longitudinal components $\mathcal{E}_y$ and $\mathcal{H}_y$ given by

(4.1b)
$$-ik\mu\,\mathcal{H}_y = \nabla_T \cdot \underset{\sim}{y}_o \times \underset{\sim}{\mathcal{E}}_T$$
$$-ik\varepsilon\,\mathcal{E}_y = \nabla_T \cdot \underset{\sim}{\mathcal{H}}_T \times \underset{\sim}{y}_o \quad ,$$

where $\nabla_T = \nabla - \underset{\sim}{y}_o \frac{\partial}{\partial y} = \underset{\sim}{x}_o \frac{\partial}{\partial x} + i\varkappa \underset{\approx}{z}_o$ and $\varepsilon = \varepsilon(y)$, $\mu = \mu(y)$. In accord with Eqs. (2.2b) the discrete mode fields $\underset{\sim}{\mathcal{E}}$ and $\underset{\sim}{\mathcal{H}}$ are to be subject to the boundary conditions

(4.2a) $$\mathcal{E}_y = \pm Z_{yy}\mathcal{H}_z \quad \text{and} \quad \mathcal{E}_z = \mp Z_{zz}\mathcal{H}_y$$

on the x-boundaries (if any) of the cross-section, and to

(4.2b) $$\mathcal{E}_x = \mp Z_{xx}\mathcal{H}_z \quad \text{and} \quad \mathcal{E}_z = \pm Z_{zz}\mathcal{H}_x$$

on the y-boundaries (if any). The impedance coefficients $Z_{\alpha\alpha}$ are prescribed constants but may be different on the x- and y-boundaries; the $\pm$ signs refer respectively to conditions at the boundary with the larger or smaller x- or y-coordinate.

For the special case wherein the cross-section is unbounded in the x-direction, there exists a set of modes independent of x that suffice for field problems in which $\partial/\partial x = 0$. Under such circumstances the vector equations (1) automatically decompose into two distinct and separate types. The mode field of one type are found to be

(4.3a)
$$\frac{\partial \mathcal{E}_x}{\partial y} = -ik\mu\, \mathcal{H}_z$$
$$\frac{\partial \mathcal{H}_z}{\partial y} = -ik\left(\varepsilon - \frac{\varkappa^2}{k^2\mu}\right) \mathcal{E}_x ,$$

or, on elimination of $\mathcal{H}_z$,

(4.3b)
$$\left(\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial y} + k^2\varepsilon - \frac{\varkappa^2}{\mu}\right) \mathcal{E}_x = 0 ,$$

where by (2b)

(4.3c)
$$\mathcal{E}_x = \mp Z_{xx}\, \mathcal{H}_z$$

on the y-boundaries. It is to be observed from Eqs. (1b) that $k\mu\, \mathcal{H}_y = \mathcal{E}_x$, and that hence Eqs. (3) characterize 'H-type' modes that possess an $\underset{\sim}{H}$- but no $\underset{\sim}{E}$- field component along the y-direction. The mode fields of the second type are correspondingly determined by

(4.4a)
$$\frac{\partial \mathcal{E}_z}{\partial y} = ik\left(\mu - \frac{\varkappa^2}{k^2\varepsilon}\right) \mathcal{H}_x$$
$$\frac{\partial \mathcal{H}_x}{\partial y} = ik\varepsilon\, \mathcal{E}_z ,$$

or, on elimination of $\mathcal{E}_z$, by

(4.4b)
$$\left(\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial y} + k^2\mu - \frac{\varkappa^2}{\varepsilon}\right) \mathcal{H}_x = 0,$$

where by (2b)

(4.4c)
$$\mathcal{E}_z = \pm Z_{zz}\, \mathcal{H}_x$$

on the y-boundaries. From Eqs. (1b) one notes that $-k\varepsilon \mathcal{E}_y = \varkappa \mathcal{H}_x$ and that hence Eqs. (4) characterize 'E-type' modes with an $\underset{\sim}{E}$- but no $\underset{\sim}{H}$- field component along the y-direction. As in the previous type, this designation is appropriate to the view of the cross-section as a terminated waveguide with transmission along y. From Eqs. (2.7) and (2.9) the possible mode solutions $\underset{\sim}{\mathcal{E}}_\alpha, \underset{\sim}{\mathcal{H}}_\alpha$ of Eqs. (3) and (4) are known in the hermitean case to possess orthogonality properties of the form

$$(4.5) \qquad \iint_S \underset{\sim}{\mathcal{E}}_\alpha \cdot \underset{\sim}{\mathcal{H}}_\beta^* \times \underset{\sim}{z}_o \, dS = 0, \qquad \varkappa_\alpha^2 \neq \varkappa_b^2,$$

where the integral is extended over the original xy guide cross-section and where the conjugate symbol$^{(*)}$ is to be omitted in the non-hermitean case.

When there is field variability along x, the vector eigenvalue problem (1) does not automatically separate into the two distinct types given in Eqs. (3) and (4). However, since the stratified region can be regarded as a uniform waveguide with homogeneous cross-section and with axis along y, the conventional mode formalism can be employed to reduce the determination of the mode fields $\underset{\sim}{\mathcal{E}} e^{i\varkappa z}$ and $\underset{\sim}{\mathcal{H}} e^{i\varkappa z}$ to a transmission-line problem for their essential variation along y. With respect to the y-axis one can express the transverse field components of a typical mode as

$$(4.6a) \qquad \begin{aligned} \underset{\sim}{\mathcal{E}}_T(\underset{\sim}{r}) &= \hat{V}(y) \; \hat{\underset{\sim}{e}}(x,z) \\ \underset{\sim}{\mathcal{H}}_T(\underset{\sim}{r}) &= \hat{I}(y) \; \hat{\underset{\sim}{h}}(x,z) \end{aligned}$$

whence by Eqs. (1b) the longitudinal components follow as

$$(4.6b) \qquad \begin{aligned} -ik\varepsilon \; \mathcal{E}_y &= \hat{I}(y) \; \nabla_T \cdot \hat{\underset{\sim}{h}} \times \underset{\sim}{y}_o \\ -ik\mu \; \mathcal{H}_y &= \hat{V}(y) \; \nabla_T \cdot \underset{\sim}{y}_o \times \hat{\underset{\sim}{e}} \, . \end{aligned}$$

The mode functions $\hat{\underset{\sim}{e}}$, $\hat{\underset{\sim}{h}}$ are to be so determined as to satisfy the specified conditions (2a) on the x-boundaries, and the mode amplitudes $\hat{V}$, $\hat{I}$ are to satisfy the conditions (2b) on the y-boundaries.

For the special case of perfectly conducting x-boundaries, $Z_{yy} = 0 = Z_{zz}$ in (2a) and the mode functions $\hat{\underset{\sim}{e}}$, $\hat{\underset{\sim}{h}}$ separate into the conventional TM (E-) and TE (H-) types with respect to the y-axis. The TM modes are defined by (cf. ref. [2])

(4.7a) $$\hat{\underset{\sim}{e}}'(x,z) = -\nabla_T \phi(x)\, e^{i\varkappa z} = \hat{\underset{\sim}{h}}' \times \underset{\sim}{y}_o$$

with

(4.7b) $$\left(\frac{d^2}{dx^2} + k_x'^2\right) \phi(x) = 0$$

subject to $\phi = 0$ on the x-boundaries. The TE modes are defined by

(4.8a) $$\hat{\underset{\sim}{h}}''(x,z) = -\nabla_T \Psi(x)\, e^{i\varkappa z} = \underset{\sim}{y}_o \times \hat{\underset{\sim}{e}}''$$

with

(4.8b) $$\left(\frac{d^2}{dx^2} + k_x''^2\right) \Psi(x) = 0$$

subject to $\partial\Psi/\partial x = 0$ on the x-boundaries. On substitution of Eqs. (6)-(8) into (1a), the amplitudes V, I for both mode types are found to obey the transmission-line equations

(4.9) $$\frac{d\hat{V}}{dy} = i\hat{\varkappa}\hat{Z}\hat{I}$$
$$\frac{d\hat{I}}{dy} = i\hat{\varkappa}\hat{Y}\hat{V},$$

where the variable propagation wave number $\varkappa$ and characteristic impedance $\hat{Z} = 1/\hat{Y}$ are given for the TM modes by

(4.10a) $$\hat{Y}' = \frac{1}{\hat{Z}'} = \frac{k}{\hat{x}}\,\varepsilon, \qquad \hat{x} = \sqrt{k^2\varepsilon\mu - k_x^2 - x^2}\,,$$

and for the TE modes by

(4.10b) $$\hat{Z}'' = \frac{1}{\hat{Y}''} = \frac{k}{\hat{x}}\,\mu, \qquad \hat{x} = \sqrt{k^2\varepsilon\mu - k_x^2 - x^2}$$

(note $k_x' = k_x'' = k_x$).

In view of (6a) the mode amplitudes $\hat{V}$, $\hat{I}$ are to satisfy in addition to Eqs. (9) the conditions (2b) on the y-boundaries. For the case wherein $\mathcal{Z}$ is constant on the y-boundaries and the x-boundaries are perfect conductors, the boundary conditions (2b) can be phrased in terms of a combination of TM and TE modes, viz:

(4.11a) $$\hat{V}'\,\hat{\underset{\sim}{e}}' + \hat{V}''\,\hat{\underset{\sim}{e}}'' = \mathcal{Z}\cdot(\hat{I}'\,\underset{\sim}{e}' + \hat{I}''\,\underset{\sim}{e}'')$$

on the y-boundaries, or, equivalently,

(4.11b) $$\begin{aligned} V' &= Z_{11}\,I' + Z_{12}\,I''\,, \\ V'' &= Z_{21}\,I' + Z_{22}\,I'' \end{aligned}$$

where

$$Z_{11} = \iint_S (\hat{\underset{\sim}{e}}'\cdot\mathcal{Z}\cdot\hat{\underset{\sim}{e}}')\,dS, \quad Z_{12} = \iint_S (\hat{\underset{\sim}{e}}'\cdot\mathcal{Z}\cdot\hat{e}'')\,dS = Z_{21}, \quad Z_{22} = \iint (\hat{\underset{\sim}{e}}''\cdot\mathcal{Z}\cdot\hat{\underset{\sim}{e}}'')\,dS.$$

A further specialization arises in the case of homogeneous isotropic y-boundaries; in this event the coupling of the TM and TE modes implied by (11) is removed since $Z_{12} = Z_{21} = 0$ and $Z_{11} = Z_{22}$.

The transmission-line equations (9) subject to (11) constitute a 'transverse resonance' problem in the xy-cross-section. Mathematically, they characterize a one-dimensional eigenvalue problem of the Sturm-Liouville type.

For the case of the TM modes, elimination of $\hat{V}'$ from (9) leads by use of (10a) to the scalar eigenvalue equation

$$\text{(4.12a)} \qquad \left(\frac{d}{dy}\,\frac{1}{\varepsilon}\,\frac{d}{dy} + k^2\mu - \frac{k_x^2+\chi^2}{\varepsilon}\right)\hat{I}'(y) = 0.$$

Elimination of $\hat{I}''$ from (9) and use of (10) leads to the scalar TE eigenvalue equations

$$\text{(4.12b)} \qquad \left(\frac{d}{dy}\,\frac{1}{\mu}\,\frac{d}{dy} + k^2\varepsilon - \frac{k_x^2+\chi^2}{\mu}\right)\hat{V}''(y) = 0.$$

For the case of imperfectly conducting and non-isotropic y-boundaries the eigenvalue equations (12a) and (12b) are coupled by the boundary conditions (11). As noted above, this coupling disappears for isotropic walls and for perfectly conducting walls. In the latter event the boundary conditions on Eqs. (12) are, respectively,

$$\text{(4.13a)} \qquad \frac{\partial \hat{I}'}{\partial y} = 0$$

and

$$\text{(4.13b)} \qquad \hat{V}'' = 0$$

on the y-boundaries. One observes that the eigenvalue problems (12a) and (12b) encountered in the case $\frac{\partial}{\partial x} = 0$ are not essentially different from those encountered in the cases (3b) and (4b), where $\partial/\partial x = 0$.

The determination of the solutions to the eigenvalue problems posed by Eqs. (7b), (8b) and Eqs. (12a), (12b) may be accomplished by the resolvent or characteristic Green's function method [3]. Since the eigenvalue equations (7b), (8b) are special cases of Eqs. (12), only the latter will be considered. The characteristic Green's function $g(y,y';\lambda)$ associated with Eqs. (12a), for example, is defined by

$$(4.14) \qquad \left(\frac{d}{dy}\,\frac{1}{\varepsilon}\,\frac{d}{dy} + k^2\mu - \frac{\lambda}{\varepsilon}\right) g(y,y';\lambda) = -\delta(y-y').$$

Let $\hat{\phi}_\alpha(y) = \hat{I}'_\alpha(y)$ denote the desired eigenfunctions normalized with weight function $1/\varepsilon(y)$ to unity over the y-interval, and let $\lambda_\alpha = k_x^2 + \varkappa_\alpha^2$ denote the desired eigenvalues. Then (cf. ref. [1]) one has

$$(4.15) \qquad \varepsilon(y')\,\delta(y-y') = -\frac{1}{2\pi i}\oint g(y,y';\lambda)\,d\lambda = \sum_\alpha \hat{\phi}_\alpha(y)\,\phi^*_\alpha(y'),$$

where the contour integral is to be extended about all singularities λ_α of g in the complex λ-plane and where the sum over α includes contributions from the complete discrete and continuous spectrum. As before the complex conjugate symbol in Eq. (15) is omitted in the non-hermitean case. Since the characteristic Green's functions defined by the inhomogeneous differential equation (14) plus appropriate boundary conditions can be explicitly determined if the functional dependence of $\varepsilon(y)$ and $\mu(y)$ is sufficiently simple, Eqs. (15) constitute a pragmatic and formal solution of the scalar eigenvalue problem as will be illustrated in another report (Part II.)

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